Impermanent Loss in a DLMM

Inventory accounting across discrete liquidity bins

Ahzam Khan

ahzam.main@gmail.com

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Abstract

Discrete liquidity market makers execute against fixed-price bins rather than a single continuous reserve curve. This note derives the marked value and impermanent loss of a one-sided liquidity position after an upward move across multiple bins. The general expression shows that only crossed liquidity contributes to divergence from holding, weighted by the difference between its execution price and the final market price. For uniform liquidity, the result reduces to a geometric sum and admits a small-bin approximation in which loss grows quadratically with the number of bins crossed.

1 Introduction and Motivation

A discrete liquidity market maker does not move along one continuous reserve curve. It trades through fixed-price bins, converting inventory one bin at a time. Impermanent loss therefore depends on the prices at which those bins execute relative to the final market price.

This note derives that inventory effect for an upward move through a one-sided position. It first treats arbitrary liquidity weights and then specializes to uniform liquidity, where the result has a closed form.

2 Notation and Bin Invariant

We begin by establishing the mathematical notation and fundamental invariant that governs DLMM bins:

Symbol	Definition
P_0	Initial market price (price of active bin index i_0)
Δ	Fractional bin step (e.g., 10^{-4} for 0.01%)
P_i	Price at bin number $i_0 + i$, defined by $P_i = P_0(1+\Delta)^i$
k	Number of bins crossed during the move ($k > 0$ upward, $k < 0$ downward)
L_i	Liquidity constant in bin i, in base-token units
(x_i, y_i)	Reserves in bin i before trade
(x_i', y_i')	Reserves in bin i after trade
α	Fraction of base in the final bin sold ($0 \le \alpha \le 1$)

Definition 1 (DLMM Bin Invariant)**.** Within each bin i, DLMM enforces a constant-sum invariant:

$$x_i + \frac{y_i}{P_i} = L_i \tag{2.1}$$

Explanation: This invariant means the total base-equivalent liquidity L_i remains fixed within each bin. When trading, adding Δx_i base tokens must remove $P_i \cdot \Delta x_i$ quote tokens, and vice versa. This is fundamentally different from the constant-product invariant $(x \cdot y = k)$ used in traditional AMMs like Uniswap v2.

The invariant can be rearranged to express the relationship between base and quote tokens in value terms:

$$P_i \cdot x_i + y_i = P_i \cdot L_i \tag{2.2}$$

This form will be useful later as it directly represents the total value of assets in a bin, expressed in terms of the quote token.

3 Initial Portfolio Value (V_0)

First, we calculate the initial value of a liquidity provider's position across all bins. The LP holds reserves (x_i, y_i) across various bins.

Valuing everything in quote units (by multiplying each base token by the initial price P_0):

$$\begin{aligned} V_0 &= \sum_i \left(\underbrace{P_0 x_i}_{\text{base value}} + \underbrace{y_i}_{\text{quote}} \right) && (3.1) \\ &= P_0 \sum_i x_i + \sum_i y_i && (3.2) \end{aligned}$$

Explanation: This represents the total value of the LP's position at the initial price P_0. We convert all base tokens to their quote equivalent (by multiplying by P_0) and add the existing quote tokens. This initial value serves as the baseline for our impermanent loss calculation.

4 Bin Dynamics During an Upward Price Move ($k > 0$)

When the price rises across k full bins plus a partial fill of the $(i_0 + k)$th bin, we need to track how the reserves in each bin change. We'll analyze three groups of bins separately: fully consumed bins, the partially consumed active bin, and bins above the active bin.

4.1 Bins Fully Consumed ($i = i_0, \ldots, i_0 + k - 1$)

For these bins, all base tokens x_i are sold at price P_i. After the trade:

$$\begin{aligned} x_i' &= 0 && (4.1) \\ y_i' &= y_i + P_i \cdot x_i && (4.2) \end{aligned}$$

Using the bin invariant $x_i + \frac{y_i}{P_i} = L_i$, we can multiply through by P_i to get $P_i \cdot x_i + y_i = P_i \cdot L_i$.

Therefore:

$$x_i' = 0 \tag{4.3}$$

$$y'_i = P_i \cdot L_i \tag{4.4}$$

Explanation: When price rises through a bin, all base tokens in that bin are sold and converted to quote tokens. The final amount of quote tokens equals the bin's total value, which is $P_i \cdot L_i$ according to our invariant.

4.2 Partially Consumed Bin $i^* = i_0 + k$

This is the active bin where the price movement stops. Let α be the fraction of base tokens sold in this bin. In the one-sided upward-moving position considered here, the bin begins with $x_{i^*} = L_{i^*}$ and $y_{i^*} = 0$.

After selling αx_{i^*} base tokens:

$$x'_{i^*} = x_{i^*} - \alpha x_{i^*} = (1-\alpha)x_{i^*} \tag{4.5}$$

$$y'_{i^*} = y_{i^*} + P_{i^*} \cdot (\alpha x_{i^*}) = y_{i^*} + \alpha P_{i^*} \cdot x_{i^*} \tag{4.6}$$

Using the one-sided starting reserves:

$$x'_{i^*} = (1-\alpha) \cdot L_{i^*} \tag{4.7}$$

$$y'_{i^*} = \alpha P_{i^*} \cdot L_{i^*} \tag{4.8}$$

Explanation: In the active bin, only a portion α of the base tokens are sold. The remaining base tokens are $(1-\alpha)$ of the original amount, and the quote tokens increase by the value of the sold base tokens. The invariant allows us to express these in terms of the bin's liquidity constant L_{i^*}.

4.3 Bins Above i^* $(i > i^*)$

These bins remain unaffected by the price movement, so:

$$x'_i = x_i = L_i \tag{4.9}$$

$$y'_i = y_i = 0 \tag{4.10}$$

Explanation: Typically, LPs load bins above the current price with only base tokens (to be sold if price rises), so we assume $y_i = 0$ here. Since the price hasn't reached these bins yet, they remain unchanged.

5 Final Portfolio Value (V_1)

Key Insight: After the price movement, the market price is $P_f = P_{i^*} = P_0(1+\Delta)^k$. To calculate the portfolio's final value, we need to account for all assets across all bins, properly valued at the new market price.

The final portfolio value must include: 1. Quote tokens from all bins below the active bin $(i < i^*)$ 2. Quote tokens in the active bin $(i = i^*)$ 3. Remaining base tokens in the active bin and above $(i \geq i^*)$, valued at the final price P_f

$$V_1 = \underbrace{\sum_{i<i^*} y'_i}_{\text{all quote below}} + \underbrace{\alpha P_f L_{i^*}}_{\text{quote from the active bin}} + \underbrace{P_f \sum_{i\geq i^*} x'_i}_{\text{remaining base}} \tag{5.1}$$

$$(5.2)$$

Explanation: This formula correctly accounts for all assets after the price movement. The first term captures all quote tokens in bins that were fully consumed. The second term represents the quote tokens in the partially consumed active bin. The third term values all remaining base tokens at the new market price P_f.

Substituting the reserve expressions from sections 4.1-4.3:

$$V_1 = \sum_{i<i^*} P_i L_i + \alpha P_f L_{i^*} + P_f \left[(1-\alpha) L_{i^*} + \sum_{i>i^*} L_i \right] \tag{5.3}$$

$$= \sum_{i<i^*} P_i L_i + \alpha P_f L_{i^*} + P_f (1-\alpha) L_{i^*} + P_f \sum_{i>i^*} L_i \tag{5.4}$$

$$\tag{5.5}$$

Notice that $\alpha P_f L_{i^*} + P_f (1-\alpha) L_{i^*} = P_f L_{i^*}$, so we can simplify:

$$V_1 = \sum_{i<i^*} P_i L_i + P_f L_{i^*} + P_f \sum_{i>i^*} L_i \tag{5.6}$$

$$= \sum_{i<i^*} P_i L_i + P_f \left[L_{i^*} + \sum_{i>i^*} L_i \right] \tag{5.7}$$

$$\tag{5.8}$$

Expanding the range of the first sum and substituting $i^* = i_0 + k$:

$$V_1 = \sum_{j=0}^{k-1} P_{i_0+j} L_{i_0+j} + P_f \left[L_{i^*} + \sum_{j=k+1}^{N_u} L_{i_0+j} \right] \tag{5.9}$$

$$\tag{5.10}$$

Where N_u is the highest bin index the LP provided liquidity to.

Further substituting $P_{i_0+j} = P_0(1+\Delta)^j$ and $P_f = P_0(1+\Delta)^k$:

$$V_1 = \sum_{j=0}^{k-1} P_0 (1+\Delta)^j L_{i_0+j} + P_0 (1+\Delta)^k \left[L_{i^*} + \sum_{j=k+1}^{N_u} L_{i_0+j} \right] \tag{5.11}$$

$$= P_0 \sum_{j=0}^{k-1} (1+\Delta)^j L_{i_0+j} + P_0 (1+\Delta)^k \left[L_{i_0+k} + \sum_{j=k+1}^{N_u} L_{i_0+j} \right] \tag{5.12}$$

$$\tag{5.13}$$

Final Expression: This gives our complete formula for the final portfolio value:

$$V_1 = P_0 \sum_{j=0}^{k-1} (1+\Delta)^j L_{i_0+j} + P_0 (1+\Delta)^k \left[L_{i_0+k} + \sum_{j=k+1}^{N_u} L_{i_0+j} \right] \tag{5.14}$$

5.1 Uniform Liquidity Distribution Simplification

If every bin has the same liquidity constant ($L_i = L$ for all i), we can simplify the expression considerably.

Let $r = 1 + \Delta$ for brevity:

$$V_1 = P_0 L \sum_{j=0}^{k-1} r^j + P_0 r^k L \left[1 + (N_u - k)\right] \tag{5.15}$$

$$= P_0 L \sum_{j=0}^{k-1} r^j + P_0 r^k L(N_u - k + 1) \tag{5.16}$$

$$\tag{5.17}$$

For the geometric series $\sum_{j=0}^{k-1} r^j$, we use the formula $\sum_{j=0}^{n-1} r^j = \frac{r^n - 1}{r-1}$:

$$\sum_{j=0}^{k-1} r^j = \frac{r^k - 1}{r - 1} = \frac{r^k - 1}{\Delta} \tag{5.18}$$

$$\tag{5.19}$$

Therefore:

$$V_1 = P_0 L \frac{r^k - 1}{\Delta} + P_0 r^k L(N_u - k + 1) \tag{5.20}$$

$$= P_0 L \left[\frac{r^k - 1}{\Delta} + r^k (N_u - k + 1)\right] \tag{5.21}$$

$$\tag{5.22}$$

Small-Δ Approximation: For finely spaced bins ($\Delta \ll 1$), we can use the approximation:

$$r^k \approx 1 + k\Delta + \frac{1}{2} k(k-1)\Delta^2 + O(\Delta^3) \tag{5.23}$$

Substituting and keeping terms up to $O(\Delta)$:

$$V_1 \approx P_0 L \left[\frac{k\Delta + \frac{1}{2}k(k-1)\Delta^2}{\Delta} + (1 + k\Delta)(N_u - k + 1)\right] \tag{5.24}$$

$$\approx P_0 L \left[k + \frac{1}{2}k(k-1)\Delta + (N_u - k + 1) + k\Delta(N_u - k + 1)\right] \tag{5.25}$$

$$\approx P_0 L \left[(N_u + 1) + \Delta \left(\frac{1}{2}k(k-1) + k(N_u - k + 1)\right)\right] + O(\Delta^2) \tag{5.26}$$

$$\tag{5.27}$$

Explanation: For fixed k, the position value approaches $P_0 L(N_u + 1)$ as Δ approaches zero. The first-order term contains both the value realized by crossed bins and the higher mark applied to inventory that remains in base.

6 HODL Baseline Value

To calculate impermanent loss, we need to compare the realized portfolio value with the "HODL" scenario, where the LP simply holds their initial assets through the price change.

If the LP had simply held their initial inventory (x_i, y_i) while the price changed to P_f, their wealth would be:

$$\text{HODL}_1 = P_f \sum_i x_i + \sum_i y_i \tag{6.1}$$

$$= P_0(1+\Delta)^k \sum_i x_i + \sum_i y_i \tag{6.2}$$

Explanation: In the HODL scenario, the LP doesn't participate in trading. Their base tokens are now valued at the new price $P_f = P_0(1+\Delta)^k$, and their quote tokens remain unchanged.

7 Impermanent Loss Definition and Calculation

Impermanent loss (IL) is defined as the relative difference between the realized LP portfolio value and the HODL alternative:

$$\mathrm{IL} = \frac{V_1}{\mathrm{HODL}_1} - 1 \tag{7.1}$$

Explanation: This measures how much value the LP loses (or gains) by providing liquidity compared to simply holding their assets. A negative value represents a loss.

8 Worked Example: Uniform One-Sided Deposit

Let's consider a specific case where: - Each bin has the same liquidity: $L_i = L$ for $i_0 \le i \le i_0 + N_u$ - The LP deposited only base tokens (single-sided liquidity) - The active bin is fully consumed: $\alpha = 1$

Starting with the uniform liquidity formula for V_1:

$$V_1 = P_0 L \left[\frac{r^k - 1}{\Delta} + r^k (N_u - k + 1) \right] \tag{8.1}$$

For the HODL baseline, since the LP initially provided only base tokens:

$$\mathrm{HODL}_1 = P_f \sum_i x_i + \sum_i y_i \tag{8.2}$$

$$= P_f (N_u + 1) L \tag{8.3}$$

$$= P_0 r^k (N_u + 1) L \tag{8.4}$$

Now we can compute the impermanent loss:

$$\mathrm{IL}(k) = \frac{V_1}{\mathrm{HODL}_1} - 1 \tag{8.5}$$

$$= \frac{P_0 L \left[\frac{r^k - 1}{\Delta} + r^k (N_u - k + 1) \right]}{P_0 r^k (N_u + 1) L} - 1 \tag{8.6}$$

$$= \frac{\frac{r^k - 1}{\Delta} + r^k (N_u - k + 1)}{r^k (N_u + 1)} - 1 \tag{8.7}$$

$$= \frac{\frac{r^k - 1}{\Delta} + r^k (N_u - k + 1) - r^k (N_u + 1)}{r^k (N_u + 1)} \tag{8.8}$$

$$= \frac{\frac{r^k - 1}{\Delta} - r^k k}{r^k (N_u + 1)} \tag{8.9}$$

$$\tag{8.10}$$

For small Δ, using our previous approximation $r^k \approx 1 + k\Delta + \frac{1}{2}k(k-1)\Delta^2 + O(\Delta^3)$:

$$\mathrm{IL}(k) \approx \frac{\frac{k\Delta+\frac{1}{2}k(k-1)\Delta^2}{\Delta} - (1+k\Delta)k}{(1+k\Delta)(N_u+1)} \tag{8.11}$$

$$\approx \frac{k + \frac{1}{2}k(k-1)\Delta - k - k^2\Delta}{(N_u+1)(1+k\Delta)} \tag{8.12}$$

$$\approx \frac{\frac{1}{2}k(k-1)\Delta - k^2\Delta}{(N_u+1)} \cdot \frac{1}{1+k\Delta} \tag{8.13}$$

$$\approx \frac{k\Delta(\frac{1}{2}(k-1) - k)}{(N_u+1)} \cdot (1 - k\Delta + O(\Delta^2)) \tag{8.14}$$

$$\approx \frac{-\frac{1}{2}k(k+1)\Delta}{(N_u+1)} + O(\Delta^2) \tag{8.15}$$

$$\tag{8.16}$$

Explanation: This shows that for small bin sizes ($\Delta \ll 1$), the impermanent loss is approximately proportional to $-\frac{k(k+1)\Delta}{2(N_u+1)}$. As expected, the loss grows quadratically with the number of bins crossed (k).

9 Extensions and Practical Notes

9.1 Dynamic Fees

In practical DLMM implementations, fees are charged on trades and distributed to LPs. To account for fee income:

1. Calculate fee income per bin based on the trade volume and fee rates
2. Add this fee income to the final portfolio value V_1 before calculating IL

Many DLMMs use dynamic fees that scale with volatility, which can significantly offset impermanent loss during volatile periods.

9.2 Non-uniform Liquidity Distributions

When liquidity is distributed non-uniformly across bins (e.g., Gaussian or exponential distributions), the sums in the V_1 formula become weighted geometric series. These can still be evaluated analytically for many common distributions or computed numerically.

9.3 Downward Price Moves

For price decreases ($k < 0$), the analysis is symmetric: swap the roles of base and quote tokens and repeat the algebra. The formulation remains essentially the same, with some sign changes to reflect the direction.

10 Conclusion

This derivation shows how impermanent loss arises from discrete bin execution. The main results are:

1. Each crossed bin realizes quote at its fixed bin price, below the final price of an upward move
2. The magnitude of IL depends on the bin size (Δ), number of bins crossed (k), and liquidity distribution
3. For fixed k and small Δ, uniform-liquidity loss is approximately $-k(k+1)\Delta/[2(N_u+1)]$
4. Fees add to LP value but must be modelled from the realized path and active liquidity

The exact weighted-bin expression should be used for arbitrary liquidity distributions and larger price moves.